Diagram Chasing and the Snake Lemma

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February 5, 2026

1 Introduction

In this short note, we provide full proofs of some standard homological algebra results. As the name of this document suggests, we will be employing the method of diagram chasing. The results are fairly "easy" yet writing up a rigorous proof is much more painful than it seems. Lastly, we have chosen the standard wording for each lemma (i.e. Wikipedia's wording) but note there are sometimes "different" lemmas with exact same names (though they are all very close and only differ in some assumptions).

2 The Four Lemma

Lemma 2.1. *(The Four Lemma) Let the following be a commutative diagram where the rows are exact:*

$$\begin{array}{ccccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \xrightarrow{g_1} & B' & \xrightarrow{g_2} & C' & \xrightarrow{g_3} & D' \end{array}$$

1. *If α is surjective, and β and δ are injective, then γ is injective;*

2. *If δ is injective, and α and γ are surjective, then β is surjective.*

Proof. (1) Suppose α is surjective, and β and δ are injective. We need to show γ is injective. To this end, let $x \in C$ be such that $\gamma(x) = 0$. This means that $0 = g_3(\gamma(x)) = \delta(f_3(x))$. But δ is injective, so $f_3(x) = 0$. Since $\ker(f_3) = \operatorname{im}(f_2)$, we have some $b \in B$ such that $f_2(b) = x$. This means $0 = \gamma(x) = \gamma(f_2(b)) = g_2(\beta(b))$. This means that $\beta(b) \in \operatorname{im}(g_1)$, i.e. we have some $a' \in A'$ such that $g_1(a') = \beta(b)$. Since α is surjective, we have some $a \in A$ such that $\alpha(a) = a'$, so we have $\beta(f_1(a)) = g_1(\alpha(a)) = g_1(a') = \beta(b)$. But β is injective, so $f_1(a) = b$. Applying f_2 we have $0 = f_2(f_1(a)) = f_2(b) = x$ as desired.

(2) Let $b' \in B'$. We need to find some $b \in B$ such that $\beta(b) = b'$. Consider $g_2(b')$. Since γ is surjective, we have some $c \in C$ such that $\gamma(c) = g_2(b')$. We see that $0 = g_3(g_2(b')) = g_3(\gamma(c)) = \delta(f_3(c))$, but δ is injective, so $f_3(c) = 0$. Hence, we have some $b \in B$ such that $f_2(b) = c$. Applying γ we have $g_2(\beta(b)) = \gamma(f_2(b)) = \gamma(c) = g_2(b')$ which means $g_2(\beta(b) - b') = 0$, i.e. we have some $a' \in A'$ such that $g_1(a') = \beta(b) - b'$. Since α is surjective, we have some $a \in A$ such that $\alpha(a) = a'$, i.e. $\beta(f_1(a)) = g_1(\alpha(a)) = g_1(a') = \beta(b) - b'$, so $\beta(b - f_1(a)) = b'$, i.e. β is surjective as desired. □

3 The Five Lemma

Lemma 3.1. *(The Five Lemma) Let the following be a commutative diagram where the rows are exact:*

$$\begin{array}{ccccccccc}
A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D & \xrightarrow{f_4} & E \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \eta} \\
A' & \xrightarrow{g_1} & B' & \xrightarrow{g_2} & C' & \xrightarrow{g_3} & D' & \xrightarrow{g_4} & E'
\end{array}$$

If β and δ are isomorphisms, α is surjective, and η is injective, then γ is an isomorphism.

Proof. Applying Part (1) of the Four Lemma to the sub-diagram on A, B, C, D implies that γ is injective. Applying Part (2) of the Four Lemma to the sub-diagram on B, C, D, E (where η plays the role of the injective map on the far right) implies that γ is surjective. $\square$

Lemma 3.2. *(Short Five Lemma) Let the following be a commutative diagram where the rows are short exact sequences:*

$$\begin{array}{ccccccccc}
0 & \xrightarrow{f_1} & A & \xrightarrow{f_2} & B & \xrightarrow{f_3} & C & \xrightarrow{f_4} & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \xrightarrow{g_1} & A' & \xrightarrow{g_2} & B' & \xrightarrow{g_3} & C' & \xrightarrow{g_4} & 0
\end{array}$$

If α and γ are isomorphisms, then β is an isomorphism.

Proof. This is an immediate consequence of the Five Lemma. Since the rows are short exact, the objects at the ends of the five-term sequence are 0. The vertical maps $0 \to 0$ are trivially isomorphisms. Thus, by the Five Lemma, if the maps on the neighbors of B (namely α and γ) are isomorphisms, then β must be an isomorphism. $\square$

4 The Snake Lemma

Lemma 4.1. *(The Snake Lemma) Let the following be a commutative diagram with exact rows:*

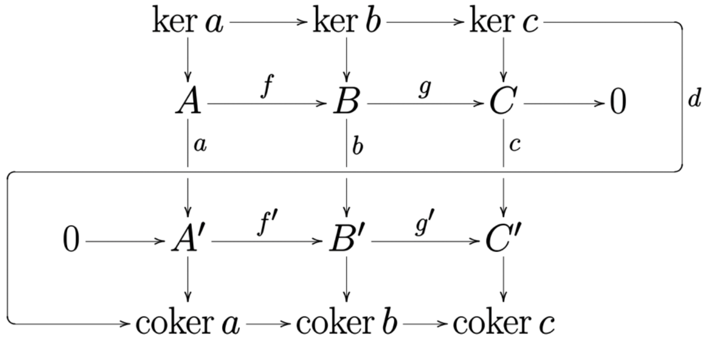
$$\begin{array}{ccccccc}
 & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & &
\end{array}$$

Then there exists an exact sequence of kernels and cokernels:

$$\ker(f) \to \ker(a) \to \ker(b) \to \ker(c) \xrightarrow{d} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c) \to \operatorname{coker}(g')$$

where d is a homomorphism, known as the connecting homomorphism.

Proof. Before we prove this result, we claim the following diagram from Wikipedia should be enough of a clue as to make the proof "easy" to guess yet still rather annoying to fully write out:



Let us establish the exactness of the sequence $\ker(f) \to \ker(a) \to \ker(b) \to \ker(c)$. First, consider the map $\ker(f) \to \ker(a)$. Let $x \in \ker(f)$, so $f(x) = 0$. By commutativity, we have $f'(a(x)) = b(f(x)) = b(0) = 0$. Since the bottom row is exact at A', f' is injective, so $a(x) = 0$. Thus $x \in \ker(a)$, i.e., $\ker(f) \subseteq \ker(a)$, so the first map is simply the inclusion. Next, we look at the map $\ker(a) \to \ker(b)$. Let $x \in \ker(a)$. We have $a(x) = 0$, so by commutativity $b(f(x)) = f'(a(x)) = f'(0) = 0$. This means $f(x) \in \ker(b)$. Thus, the map restricts to $f|_{\ker(a)} : \ker(a) \to \ker(b)$. We now check for exactness. The kernel of this restricted map is $\{x \in \ker(a) : f(x) = 0\}$, which is exactly $\ker(f) \cap \ker(a)$. Since we showed $\ker(f) \subseteq \ker(a)$, this intersection is just $\ker(f)$, which is exactly the image of the inclusion map from the previous step. Finally, we consider the map $\ker(b) \to \ker(c)$. Let $x \in \ker(b)$. We have $b(x) = 0$, so by commutativity $c(g(x)) = g'(b(x)) = g'(0) = 0$. This means $g(x) \in \ker(c)$. Thus, the map restricts to $g|_{\ker(b)} : \ker(b) \to \ker(c)$. Now we check for exactness. First, we show im $\subseteq$ ker. For any $y \in \ker(a)$, $g(f(y)) = 0$ by the exactness of the top row, so $f(\ker(a)) \subseteq \ker(g) \cap \ker(b) \subseteq \ker(g|_{\ker(b)})$. Second, we show ker $\subseteq$ im. Let $x \in \ker(b)$ such that $g(x) = 0$ (i.e., x is in the kernel of the restricted map). By the exactness of the top row at B, since $x \in \ker(g)$, there exists some $y \in A$ such that $f(y) = x$. We must show $y \in \ker(a)$. Observe that $f'(a(y)) = b(f(y)) = b(x)$. Since $x \in \ker(b)$, $b(x) = 0$, so $f'(a(y)) = 0$. Because f' is injective, $a(y) = 0$, which implies $y \in \ker(a)$. Thus x is in the image of $f|_{\ker(a)}$, as desired.

Let us take care of the exactness of $\text{coker}(a) \to \text{coker}(b) \to \text{coker}(c) \to \text{coker}(g')$ now. First, we define $\overline{f'} : \text{coker}(a) \to \text{coker}(b)$ by $x + \text{im}(a) \mapsto f'(x) + \text{im}(b)$. To see this is well-defined, suppose $x, y \in A'$ represent the same class, i.e., $x - y \in \text{im}(a)$. Then $x - y = a(z)$ for some $z \in A$. Thus $f'(x) - f'(y) = f'(x - y) = f'(a(z)) = b(f(z)) \in \text{im}(b)$. So the images are equivalent in $\text{coker}(b)$. Next, we define $\overline{g'} : \text{coker}(b) \to \text{coker}(c)$ by $x + \text{im}(b) \mapsto g'(x) + \text{im}(c)$. This is well-defined because if $x - y \in \text{im}(b)$, then $x - y = b(z)$ for some $z \in B$. Thus $g'(x) - g'(y) = g'(b(z)) = c(g(z)) \in \text{im}(c)$. Now we check for exactness at $\text{coker}(b)$. Clearly $\text{im}(\overline{f'}) \subseteq \ker(\overline{g'})$ because $g' \circ f' = 0$. For the reverse inclusion, let $y \in B'$ represent a class in $\ker(\overline{g'})$. This means $g'(y) \in \text{im}(c)$, so $g'(y) = c(z)$ for some $z \in C$. Since the top row is exact at C (i.e. g is surjective), there exists $w \in B$ such that $g(w) = z$. Observe that $g'(y) = c(g(w)) = g'(b(w))$. This implies $g'(y - b(w)) = 0$. By the exactness of the bottom row at B', $y - b(w) = f'(x)$ for some $x \in A'$. In terms of cokernels, $y + \text{im}(b) = (y - b(w)) + \text{im}(b) = f'(x) + \text{im}(b) = \overline{f'}(x + \text{im}(a))$. Thus $\ker(\overline{g'}) \subseteq \text{im}(\overline{f'})$. Lastly, let us make a map $\text{coker}(c) \to \text{coker}(g')$ (where $\text{coker}(g') = C'/\text{im}(g')$). Define the map by $z + \text{im}(c) \mapsto z + \text{im}(g')$. This is well-defined because if $z \in \text{im}(c)$, then $z = c(x)$ for some $x \in C$. Since g is surjective (top row), $x = g(y)$ for some $y \in B$. Then $z = c(g(y)) = g'(b(y)) \in \text{im}(g')$. For exactness at $\text{coker}(c)$: The kernel of this projection is precisely $\{z + \text{im}(c) : z \in \text{im}(g')\}$, which is exactly the image of $\overline{g'}$.

Finally, let us consider the connecting map $d : \ker(c) \to \text{coker}(a)$. Let $x \in \ker(c)$. Since g is surjective,

there exists $y \in B$ such that $g(y) = x$. Consider $b(y) \in B'$. By commutativity, $g'(b(y)) = c(g(y)) = c(x) = 0$. Thus $b(y) \in \ker(g')$. By exactness of the bottom row, $\ker(g') = \operatorname{im}(f')$. Since f' is injective, there exists a unique $z \in A'$ such that $f'(z) = b(y)$. We define $d(x) = z + \operatorname{im}(a) \in \operatorname{coker}(a)$. We must check this is well-defined. Suppose we chose a different lift y' such that $g(y') = x$. Then $g(y - y') = x - x = 0$, so $y - y' \in \ker(g) = \operatorname{im}(f)$. Thus $y - y' = f(w)$ for some $w \in A$. Let z and z' be the elements in A' associated with y and y' respectively. Then $f'(z - z') = f'(z) - f'(z') = b(y) - b(y') = b(y - y') = b(f(w))$. By commutativity, $b(f(w)) = f'(a(w))$. Since f' is injective, $z - z' = a(w)$, which means $z - z' \in \operatorname{im}(a)$. Thus z and z' represent the same class in $\operatorname{coker}(a)$, so the map is well-defined. Moreover, to see that d is an R-module homomorphism, let $x_1, x_2 \in \ker(c)$ and $r \in R$. Choose lifts $y_1, y_2 \in B$ such that $g(y_i) = x_i$. Then $g(y_1 + y_2) = x_1 + x_2$, so $y_1 + y_2$ is a valid lift for the sum. Let z_1, z_2 be the unique elements in A' such that $f'(z_i) = b(y_i)$. Then $f'(z_1+z_2) = f'(z_1)+f'(z_2) = b(y_1)+b(y_2) = b(y_1+y_2)$. Thus, the element in A' associated with x_1+x_2 is exactly z_1+z_2. Projecting to the cokernel, $d(x_1+x_2) = (z_1+z_2)+\operatorname{im}(a) = d(x_1)+d(x_2)$. Similarly, for scalar multiplication, since g is R-linear, $g(ry_1) = rg(y_1) = rx_1$, so ry_1 is a valid lift for rx_1. Then $b(ry_1) = rb(y_1) = rf'(z_1) = f'(rz_1)$ by the linearity of b and f'. Thus the unique element in A' associated with rx_1 is rz_1. Projecting to the cokernel, $d(rx_1) = rz_1+\operatorname{im}(a) = r(z_1+\operatorname{im}(a)) = rd(x_1)$.
Next, we establish exactness at $\ker(c)$. We must show $\operatorname{im}(\operatorname{res} g) = \ker(d)$. Let $y \in \ker(b)$. Then $g(y) \in \ker(c)$. To compute $d(g(y))$, we lift $g(y)$ to y. Then we push to $b(y)$, which is 0 since $y \in \ker(b)$. The unique $z \in A'$ such that $f'(z) = 0$ is $z = 0$ since f' is injective. Thus $d(g(y)) = 0 + \operatorname{im}(a) = 0$. Conversely, suppose $x \in \ker(c)$ and $d(x) = 0$. Lift x to $y \in B$. By definition, $d(x)$ is the class of z where $f'(z) = b(y)$. If $d(x) = 0$, then $z \in \operatorname{im}(a)$, so $z = a(w)$ for some $w \in A$. Observe that $b(y) = f'(z) = f'(a(w)) = b(f(w))$. Thus $b(y - f(w)) = 0$, so $y - f(w) \in \ker(b)$. Applying g, we get $g(y - f(w)) = g(y) - g(f(w)) = x - 0 = x$. Thus x is the image of an element in $\ker(b)$.
Finally, we establish exactness at $\operatorname{coker}(a)$. We must show $\operatorname{im}(d) = \ker(\overline{f'})$. Let $z + \operatorname{im}(a) \in \operatorname{im}(d)$. Then z was obtained from some $y \in B$ via $f'(z) = b(y)$. Then $\overline{f'}(z + \operatorname{im}(a)) = f'(z) + \operatorname{im}(b) = b(y) + \operatorname{im}(b) = 0$. Conversely, let $z + \operatorname{im}(a) \in \ker(\overline{f'})$. Then $f'(z) \in \operatorname{im}(b)$, so $f'(z) = b(y)$ for some $y \in B$. We claim $g(y) \in \ker(c)$. Indeed, $c(g(y)) = g'(b(y)) = g'(f'(z)) = 0$ by the exactness of the bottom row. Thus we can compute $d(g(y))$. We lift $g(y)$ to y. We push to $b(y)$. We pull back via f' to get exactly z. Thus $d(g(y)) = z + \operatorname{im}(a)$, so the element is in the image of d. □

5 The Nine, Sixteen, Twenty-Five, and so on Lemma

Lemma 5.1. *(The Nine Lemma) Consider the following commutative diagram with exact columns:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma_1} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma_2} & & \\
0 & \longrightarrow & A_3 & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

1. *If the two bottom rows are exact, then the top row is exact.*

2. *If the two top rows are exact, then the bottom row is exact.*
3. *If the top and bottom rows are exact, and the composition of the middle row is zero ($g_2 \circ f_2 = 0$), then the middle row is exact.*

Proof. (1) Assume the bottom two rows are exact. We apply the Snake Lemma to the map of short exact sequences from the middle row ($0 \to A_2 \to B_2 \to C_2 \to 0$) to the bottom row ($0 \to A_3 \to B_3 \to C_3 \to 0$). Since the columns are exact, we have $\mathrm{coker}(\alpha_2) = \mathrm{coker}(\beta_2) = \mathrm{coker}(\gamma_2) = 0$. Since the middle row is exact, the first map $A_2 \to B_2$ is injective, so its kernel is 0. The Snake Lemma then yields the exact sequence:

$$0 \to \ker(\alpha_2) \to \ker(\beta_2) \to \ker(\gamma_2) \xrightarrow{\partial} \mathrm{coker}(\alpha_2) \dots$$

Substituting the known kernels (A_1, B_1, C_1) and cokernels (0), we get precisely the exact sequence:

$$0 \to A_1 \to B_1 \to C_1 \to 0$$

Thus, the top row is exact.
(2) Assume the top two rows are exact. We apply the Snake Lemma to the map of short exact sequences from the top row ($0 \to A_1 \to B_1 \to C_1 \to 0$) to the middle row ($0 \to A_2 \to B_2 \to C_2 \to 0$). Since the columns are exact, we have $\ker(\alpha_1) = \ker(\beta_1) = \ker(\gamma_1) = 0$. Since the middle row is exact, the last map $B_2 \to C_2$ is surjective, so its cokernel is 0. The Snake Lemma then yields the exact sequence:

$$\dots \ker(\gamma_1) \xrightarrow{\partial} \mathrm{coker}(\alpha_1) \to \mathrm{coker}(\beta_1) \to \mathrm{coker}(\gamma_1) \to 0$$

Substituting the known kernels (0) and cokernels (A_3, B_3, C_3), we get precisely:

$$0 \to A_3 \to B_3 \to C_3 \to 0$$

Thus, the bottom row is exact.
(3) Assume the top and bottom rows are exact and $g_2 \circ f_2 = 0$. We must show exactness at A_2, B_2, C_2. First, let us show exactness at A_2, i.e. injectivity. Let $a \in A_2$ such that $f_2(a) = 0$. Then $\beta_2(f_2(a)) = 0$. By commutativity, $f_3(\alpha_2(a)) = 0$. Since the bottom row is exact, f_3 is injective, so $\alpha_2(a) = 0$. By column exactness, $a \in \mathrm{im}(\alpha_1)$, so $a = \alpha_1(x)$ for some $x \in A_1$. Then $f_2(a) = f_2(\alpha_1(x)) = \beta_1(f_1(x)) = 0$. Since β_1 is injective, $f_1(x) = 0$. Since the top row is exact, f_1 is injective, so $x = 0$, implying $a = 0$. Now, let us show Exactness at B_2. Since $g_2 \circ f_2 = 0$, we have $\mathrm{im}(f_2) \subseteq \ker(g_2)$. We need to show $\ker(g_2) \subseteq \mathrm{im}(f_2)$. Let $b \in B_2$ such that $g_2(b) = 0$. Then $\gamma_2(g_2(b)) = 0$. By commutativity, $g_3(\beta_2(b)) = 0$. Since the bottom row is exact, $\beta_2(b) \in \mathrm{im}(f_3)$. Let $a' \in A_3$ such that $f_3(a') = \beta_2(b)$. Since the map $\alpha_2 : A_2 \to A_3$ is surjective (column exactness), there exists $a \in A_2$ such that $\alpha_2(a) = a'$. Consider the element $b - f_2(a)$. Applying β_2, we get $\beta_2(b - f_2(a)) = \beta_2(b) - \beta_2(f_2(a)) = \beta_2(b) - f_3(\alpha_2(a)) = \beta_2(b) - f_3(a') = 0$. Thus $b - f_2(a) \in \ker(\beta_2)$. By column exactness, $\ker(\beta_2) = \mathrm{im}(\beta_1)$, so there exists $b_1 \in B_1$ such that $\beta_1(b_1) = b - f_2(a)$. Now apply g_2 to this equality: $g_2(\beta_1(b_1)) = g_2(b - f_2(a))$. The LHS is $\gamma_1(g_1(b_1))$ by commutativity. The RHS is $g_2(b) - g_2(f_2(a)) = 0 - 0 = 0$ (since $b \in \ker g_2$ and $g_2 f_2 = 0$). Thus $\gamma_1(g_1(b_1)) = 0$. Since γ_1 is injective, $g_1(b_1) = 0$. Since the top row is exact, $b_1 \in \mathrm{im}(f_1)$, so $b_1 = f_1(a_1)$ for some $a_1 \in A_1$. Substituting back: $b - f_2(a) = \beta_1(f_1(a_1)) = f_2(\alpha_1(a_1))$. Rearranging, $b = f_2(a + \alpha_1(a_1))$, so $b \in \mathrm{im}(f_2)$. Lastly, let us Exactness at C_2, i.e. surjectivity. Let $c \in C_2$. Then $\gamma_2(c) \in C_3$. Since the bottom row is exact, g_3 is surjective, so there exists $b' \in B_3$ such that $g_3(b') = \gamma_2(c)$. Since β_2 is surjective, lift b' to $b \in B_2$ such that $\beta_2(b) = b'$. Consider $g_2(b)$. Map it down: $\gamma_2(g_2(b)) = g_3(\beta_2(b)) = g_3(b') = \gamma_2(c)$. Thus $\gamma_2(g_2(b) - c) = 0$. By column exactness, $g_2(b) - c \in$

$\ker(\gamma_2) = \operatorname{im}(\gamma_1)$. So $g_2(b) - c = \gamma_1(c_1)$ for some $c_1 \in C_1$. Since the top row is exact, g_1 is surjective, so $c_1 = g_1(b_1)$ for some $b_1 \in B_1$. Then $c = g_2(b) - \gamma_1(g_1(b_1)) = g_2(b) - g_2(\beta_1(b_1)) = g_2(b - \beta_1(b_1))$. Thus $c \in \operatorname{im}(g_2)$. □

Aside 5.2. By the symmetry of the diagram, the "transpose" of these statements is also true. Exchanging the roles of "row" and "column" in (1), (2), and (3) yields three additional valid criteria for the exactness of the middle column.

Challenge: Generalize and extend this result to case of 4 short exact sequences with exact columns, then 5 short exact sequences with exact columns, and so. You should do this inductively using the Nine Lemma NOT via diagram chasing.
Challenge: Package the a short exact sequence as a single "object" and somehow rationalize all of these things above as short exact sequences of these "objects". For a hint, google chain complex.